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Conjecturing is not all: Theorizing in design research by refining and connecting categorial, descriptive, and explanatory theory elements

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Abstract Although generating empirically grounded theoretical contributions has been an essential aim of design research from the beginning, the underlying methodology has been described as needing further elaboration with respect to rigor. Existing methodological elaborations on theorizing in design research have often focused on argumentative grammar for strengthening prescriptive conjectures. Beyond these, this paper argues that prescriptive theory elements require generating categorial, descriptive, and explanatory theory elements before the suggested argumentative grammars can be treated. With an example from a design research study for professional development, illustration is given of the immense work of elaborating categorial and explanatory theory elements that is needed before concise prescriptive conjectures can be made and tested.

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Conjecturing is not all: Theorizing in design research by refining and connecting categorial, descriptive, and explanatory theory elements

Susanne Prediger

1.0 Introduction

Design research is the overarching name for several research approaches (van den Akker et al., 2006; Cobb et al., 2003) that all share the main idea of combining *research-based design* (for improving learning opportunities in classrooms and teacher education) with *design-based research* (for generating empirically grounded theoretical contributions about the initiated teaching-learning processes and substantiating prescriptive principles). Design research approaches began 30 years ago, starting from early versions in subject matter education research (Artigue, 1992; Wittmann, 1995) and educational sciences (Brown, 1992). Whereas *generic educational* design research mainly focuses on design principles and content-independent design elements (van den Akker, 1999), *didactical design research* often puts the focus on content-specific aspects, for example, on hypothetical learning trajectories for a particular area of subject-matter content (Bakker, 2018; Gravemeijer & Cobb, 2006), or on substantiating generic design principles for particular areas of subject-matter content (Prediger, 2019a). This also applies to the emerging field of design research for the professional development (PD) of teachers (PD design research).

One important strand of the efforts to successively establish design research as a field of research has been to develop it from a loosely described set of design and research activities into a *consolidated methodology* with well-articulated background theories (Gravemeijer & Cobb, 2006) and well-elaborated, more rigorous ways of theorizing (Bakker, 2018; Cobb et al., 2017; Prediger, 2019a; Sandoval, 2014). In this context, *theorizing* can be defined as the methodologically controlled and logically sound ways of generating theoretical contributions through designing, conducting, and analyzing design experiments (Prediger, 2019a; similarly, Bakker, 2018). The key question of this meta-theoretical paper is: *How can we characterize and distinguish the products and processes of theorizing in design research?*

Some design researchers have described the main theoretical contributions of design research as empirically grounded prescriptive conjectures, mainly about the connection between design elements and learning outcomes (Sandoval, 2014; Cobb et al., 2017) or about sequences of steps in content-specific hypothetical learning trajectories (Bakker, 2018; Cobb et al., 2017). The existing methodological proposals for conjecturing are presented in Section 2.

While these methodological proposals about conjecturing have been highly valuable in helping make the processes of theorizing more explicit, this paper argues that conjecturing "is not all"; in other words, most conjectures in design research are prescriptive theory elements that require other theory elements to be articulated and justified. In Section 3, the structures and functions of other theory elements are explained and used for articulating the necessary constituents of conjectures, showing what stays too implicit in the meta-theoretical discourse. In the core Section 4, an example case from PD design research is presented to illustrate the interplay of the different theory elements and the importance of explicitly reflecting on constructs and explanatory theory elements before articulating and testing conjectures. Whereas a similar suggestion was made for design research on the classroom level (Prediger, 2019a), this can contribute to the emerging field of PD design research, which is even more complex than classroom design research (Lasthein Lehrmann et al., 2022; Cobb et al., 2017).

2.0 Conjecturing as an important part of theorizing

In most educational or didactical research approaches, theories convey two roles: as *frameworks* and as *outcomes* of research (Mason & Waywood, 1996). *Background theoretical frameworks* inform design decisions and the methods and perceptions in the empirical investigations of the teaching-learning processes that have been initiated. Empirical investigations aim at generating, testing, and refining *new theoretical contributions* as outcomes. In design research, both roles of theory are highly intertwined in the iterative and interactive processes of theory-guided designs and experimentation and theory-generating processes during the analysis (Cobb et al., 2003).

Since the early years, design research has been criticized for lacking methodological rigor in the empirically grounded theorizing processes, in particular a clear argumentative grammar that Kelly defined as "the logic that guides the use of a method and that supports reasoning about its data" (Kelly, 2004, p. 118). Sandoval (2014) outlined this lack of rigor as not necessarily being in the theorizing practices themselves, but in the ways they are reported and methodologically justified: "There are surely a number of researchers … who are conducting systematic design research, but we are not talking much about how we do it … These require … methodological commitments" (pp. 19-20).

In the Subsections 2.1-2.3, three often-cited proposals for methodological clarifications are reported that all focus on conjecturing about prescriptive theory elements informing the designs.

2.1 Conjectures on hypothetical learning trajectories

Confrey (2006) characterized design research as follows: "Design researchers make, test, and refine conjectures about the learning trajectory based on evidence as they go" (p. 136). Similarly, Bakker (2018) pointed to hypothetical learning trajectories as major theoretical outcomes of didactical design research, entailing content-specific prescriptions of "how students may make progress from particular starting points to intended outcomes" (p. 58), depending on successive learning opportunities offered in the learning environment. The learning trajectory informs the design and the teaching, and can also be structured as a non-linear learning content map, given that learners' learning pathways can be non-linear and deviate around the intended trajectories (Confrey, 2006).

When suggesting an argumentative grammar for how to empirically ground the hypothesized learning trajectories, Cobb et al. (2017) conceptualized learning trajectories as a sequence of intermediate learning outcomes (e.g., successive forms of emerging mathematical reasoning) that are promoted by particular design aspects in the learning environment. They suggested three necessary research practices for providing an empirical ground for the conjectured learning trajectory with its sequenced intermediate outcomes: (a) demonstrating that students would not have reached the overall learning goal but for their participation in the learning environment, (b) "documenting how each successive form of reasoning emerged as a reorganization of prior forms of reasoning" (p. 215), and (c) identifying the design features that were necessary to reach the intermediate learning outcomes.

2.2 Sandoval's conjecture mapping as a way to unpack various elements

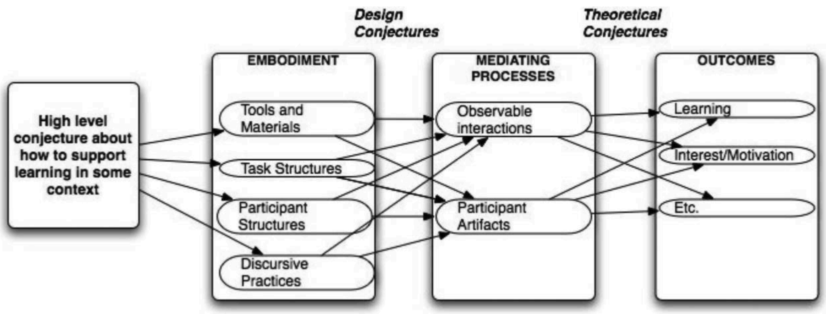
Sandoval (2014) proposed conjecture mapping "as a method for articulating the joint design and theoretical ideas embodied in a learning environment in a way that supports choices about the means for testing them" (p. 20). His conjecture map starts from a highly condensed conjecture about how to support learning in some context. This overall conjecture is then unpacked into several components: *design elements* (tools and materials, task structures, participant structures, and discourse practices) chosen to initiate certain *mediating processes* in learners (e.g., observable interaction or participant artifacts) that are assumed to generate certain *learning outcomes*.

Design conjectures are conceived as assumptions that link the design elements to some mediating processes and convey the following general form: "If learners engage in this activity (task + participant) structure with these tools, through this discursive practice, then this mediating process will emerge" (Sandoval, 2014, p. 24). Sandoval also spoke about testing conjectures for qualitative design experiment methodologies: "Testing such a conjecture requires methods that can identify whether the expected mediating process does in fact emerge and that can provide evidence to trace that process back to designed elements" (Sandoval, 2014, p. 24). In this approach, causal connections are considered in Maxwell's (2004) process-oriented conception with its focus on the mechanism, not on a statistical regularity conception of causality.

Theoretical conjectures are conceived as those that link the mediating processes to the targeted learning outcomes. Their testing might require instrumentation beyond videorecording the processes themselves, but Sandoval (2014) emphasized that tests for measuring learning outcomes can often only be developed after some design experiment cycles when the learning outcomes are specified more concisely.

Overall, the advantage of conjecture maps seems to be not only to articulate conjectures, but also first to specify the theoretically salient design elements and mediating processes. In this way, Sandoval (2014) already hinted at the high relevance of further theory elements to articulate the conjectures, as will be discussed in Section 3.

Figure 1: Generalized conjecture map guiding design research (Sandoval, 2014, p. 21)



2.3 Other kinds of argumentative grammars

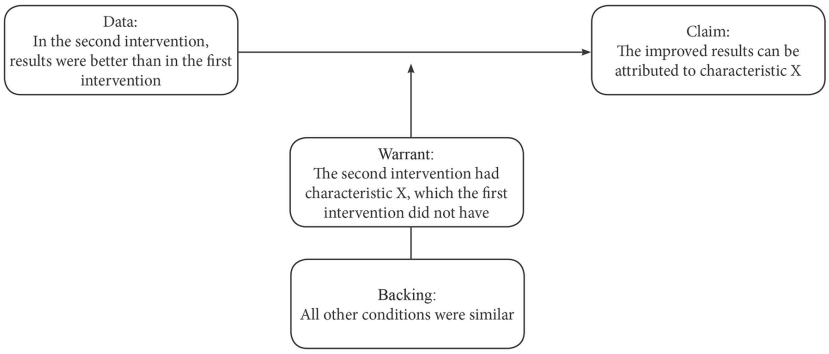
Besides the two presented argumentative grammars, Bakker (2018) reconstructs three more argumentative grammars that have frequently been applied in design research studies:

The third argumentative grammar refers to consensus in the design community (e.g., a model of teacher expertise) that is then *applied* for a particular topic in view. As this argumentative grammar alone rarely contributes new theoretical insights, it is not elaborated here, but needs to be mentioned in order to explain later how *substantiating* models goes beyond simply applying them.

The fourth argumentative grammar refers to the simple proof of existence that resonates with the first aspects of Cobb et al.'s (2017) list: The claim "It is possible for learners to achieve a learning goal [that was formerly often not achieved]" is backed up by post-test achievement data and warranted by the methodological soundness of the post-test.

The fifth and most important argumentative grammar zooms into one aspect of Sandoval's (2014) map, namely, into one design conjecture, and argues by means of carefully chosen iterative changes of only one design characteristic (Figure 2).

Figure 2: Example for an argumentative grammar belonging to small changes in the design (Bakker, 2018, p. 102)



In these briefly summarized methodological considerations of design researchers on different argumentative grammars, the methodological focus has been concentrated on prescriptive theory elements (prescribing how content sequences or instructional designs should be shaped), by warranting and backing the learning trajectories or connections between design characteristics and intended learning outcomes or mediating processes.

While these considerations are highly important, to widen the methodological considerations about theorizing to other highly relevant sub-processes, the cited methodological considerations on prescriptive conjectures *implicitly* refer to other theory elements that are explicated in the following section.

3.0 Widening the methodological considerations from prescriptive conjectures to further theory elements with different functions and structures

Following various meta-theoretical definitions of theory (Niss, 2007; Beck & Krapp, 2006), Prediger (2019a) suggested that theorizing in design research studies can be conceptualized as a process of *successively elaborating a web of intertwined theory elements with different functions*, not only prescriptive conjectures in the argumentative grammars. Subsections 3.1 and 3.2 (largely taken from Prediger, 2019a) explain not only categorial, descriptive, and explanatory functions but also normative functions of theory elements. The logic of theory elements is then used to unpack conjecturing and argumentative grammars in Subsection 3.3.

3.1 Theory elements with different structures and functions

Niss (2007) defines theory as an “organized network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about … objects, processes, situations, and phenomena” (Niss, 2007, p. 1308). His distinction of concepts and claims that are interconnected resonates with the general philosophy of science, in which concepts are called *categories* or *constructs* and the claims are called *propositions.*

	Function of the theory element	Structure of the theory element
Constructs = Categorial theory elements	Providing a language and thinking tool for perceiving and distinguishing	Conceptual structures, i.e., categories, and relations
Descriptive theory elements	Describing a certain phenomenon qualitatively or quantitatively, focused by specific categories	Propositions stating existence, categorial hierarchies, or frequencies
Explanatory theory elements	Explaining, giving causes, or identifying backgrounds	Propositions with cause-effect structure or phenomenon-background structure
Normative theory elements	Specifying and justifying aims and rationales (e.g., learning goals or process qualities)	Propositions with an aim-reason structure
Prescriptive theory elements	Purposefully acting or predicting effects	Propositions in "in order to" structure or propositions in "if-then" structure

Table 1: Five theory elements and their functions and structures (adapted from Beck & Krapp, 2006, pp. 39 ff, in Prediger, 2019a, p. 8)

Propositions can have different *functions*: A theory is a language entity in propositional or categorial form that *orders* the phenomena of a domain and *describes* the relevant features of its objects and their relations to each other; *explains* by general laws and *allows predictions* for the occurrence of phenomena. (Thiel, 1996, p. 262; similarly McKenney & Reeves, 2012).

Design research studies usually aim at complex local instruction theories, which do not address only one function but combine theory elements with different functions. Beck and Krapp (2006, pp. 39 ff) applied the classical distinctions of functions from general philosophy of science to learning and made explicit their different logical structures (see Table 1). They will be explained in the following (and illustrated by examples in the next section):

- *Constructs* have the function of providing a language and thinking tool for perceiving and distinguishing phenomena. Their logical structure is conceptual, which means that descriptive elements usually consist of constructs and their relations. Many researchers in mathematics education research have emphasized the relevance of categories or constructs for a theory's descriptive and explanatory power (e.g., Niss, 2007, p. 1308). diSessa and Cobb (2004) emphasized: "[Theoretical constructs] enable us to discriminate between relations that are necessary and those that are contingent. They delineate classes of phenomena that are worthy of inquiry and specify how to look and what to see in order to understand them. This last characteristic—epigrammatically, "teaching us how to see." (p. 79). Both authors have additionally emphasized that it can be the invention of an important category that brings a phenomenon into a new quality of being. Methodologies for generating constructs in empirically grounded processes of data-led successive refinement have been carefully reflected, for example, in grounded theory (Strauss & Corbin, 1990). Constructs are decisive for all further theory elements, as they provide the language to describe, normatively set aims, and explain or prescribe actions in propositional theory elements:

- *Descriptive theory elements* serve to describe a certain phenomenon. They answer typical questions such as: What characterizes this area? Which phenomena and relations can occur? In which frequencies? Descriptive theory elements consist of propositions of different logical structures, for example, describing features ("M has characteristics C" or "M can be C_1, C_2, or C_3"), categorial hierarchies ("Every *x* is also *y*"), or frequencies of occurrences ("20% of teachers enact the practice P_1 and 30% P_2"). Empirical research that generates new descriptive findings must make sure that the phenomena and eventually frequencies are adequately described with validity and reliability, depending on the adequacy of the constructs (Strauss & Corbin, 1990).
- *Explanatory theory elements* serve to *explain, give causes, or identify backgrounds of described phenomena*; thus they answer questions such as "Why do teachers enact a certain practice? What might be the background?" The logical structure of related propositions can be cause-effect or phenomenon-background structures ("Phenomenon *x* occurs because of *y*" or "phenomenon *x* can be traced back to phenomenon *y*"). Empirical research that generates new explanatory theory elements requires constructs and descriptive findings and empirical evidence that the phenomena are really related to the claimed background. In qualitative research approaches, this is shown by detailed analyses in which the interplay between phenomenon and background is unpacked (e.g., by contrasting cases in interpretative methodologies; e.g., Yin, 1994; Strauss & Corbin, 1990; Maxwell, 2004). Although constructs and descriptive components are required for explanatory elements, they are sometimes generated at the same time in qualitative research. In quantitative research approaches, explanatory findings require methods that can identify explaining factors statistically, for instance, in statistical path models or regression models for testing hypotheses on potential connections.
- *Normative theory elements* serve to specify and justify aims and rationales, for example, by questions such as: Which aims shall be reached (e.g., by a certain PD activity)? In which context are they justified? Normative theory elements can refer, for example, to content learning goals but also to mediating processes (e.g., active involvement of all participating teachers) that should be reached in an activity. The logical structure of normative theory elements consists of propositions connecting the aims to reasons why the aim should be reached ("Teachers should acquire PD goal *x* because this is required for practice *y*" or "the mediating processes *z* are initiated because this has been shown to enhance *w*"). Making normative elements explicit is crucial due to their role as (sometimes implicit) components of prescriptive theory elements. Whereas the aim itself in a normative theory element cannot be

"proven" empirically, the justification of this aim can refer to explanatory theory elements and therefore have an empirical foundation.
- *Prescriptive theory elements* serve to ground purposeful acting or predict effects of a design element or structural element (such as specific access to the structure of the content). They answer to questions such as: "What can be done to reach a certain aim?" or "What could happen if a decision is taken in a specific way?" Their logical structures can be an "in-order-to" structure ("in order to reach aim *x* you are advised to do *y*") or an "if-then" structure ("if you do *y*, you could reasonably expect *y*," obviously not in a deterministic logical sense). Empirical research that generates new prescriptive findings has mostly been interventionist: in qualitative research approaches by contrasting several cases (Yin, 1994) and in quantitative research approaches, for example, by the classical design of a randomized controlled trial.

This general distinction of theory elements with their functions and logical structures helps to identify different kinds of possible theoretical contributions and can hence guide the targeted theorizing process. Since different logical structures require different empirical warrants, their distinction can also support the methodological reasoning of the researcher.

3.2 Multiple and intertwined theory elements in educational and didactical design research

Design researchers often start by formulating a problem. Articulating the problem requires *descriptive theory elements*, for example, problematizing that the reality deviates from intended learning goals or other educational aims (*normative* elements). Design researchers set out to develop or refine design principles (*prescriptive* heuristics or theory elements connecting specific options for design and acting towards the intended aims) for an orientation towards how to reach their goals. During several design experiment cycles, they iteratively develop not only practical solutions for the initial problem by particular design elements (such as a task or support means), but also descriptive and explanatory findings:

In generic educational design research (as is often conducted in departments of educational sciences without relation to particular areas of subject matter), the answers to overarching "how" questions are the focus. With the articulated logical structures of Table 1 at hand, a refined design principle can now be described as a conjecture to be tested (see Section 2), so the design principles with their prescriptive function can be further decomposed: "If you want to design <intervention X> for the <purpose/function Y> in <context Z>, then you are best advised to give <that intervention> the <characteristics A, B, and C>…, because of <arguments P, Q, and R>" (van den Akker, 1999, p. 9).

	"How" questions for theory elements on designs of teaching-learning arrangements	"What" questions for theory elements on structuring the content
Constructs = Categorial theory elements	Constructs for design principles, process qualities, characteristics of design elements	Constructs for distinguishing and relating aspects of the learning content
Normative theory elements	Which mediating process should be initiated in order to achieve later learning goals (and why)? (*mediating processes*)	What should learners learn (and why)? (*unpacked learning content goals*)
Humble prescriptive heuristics	Which design principles should be applied for which aim?	In which (still vague) learning trajectory can the learning content be structured?
Descriptive theory elements	Which situational effects can the design principles and design elements unfold in the teaching-learning pathways? And how does that relate to the intended effects?	What learning pathways do participants usually take along the intended learning trajectory? And how do they relate to the intended learning trajectory?
Explanatory theory elements	Which background do the (non-)effects of design principles and design elements have? Under which conditions of success do they have the intended effects?	What can explain the participants' typical perspectives, learning pathways, and obstacles? (e.g., which aspects are crucial for learning the next one?) What can explain differences between the intended learning trajectory and the individual learning pathways?
Refined prescriptive theory elements drawing upon other elements	*Elaborated design principles:* Which design characteristics and design elements can be applied for which intended aim? Which explanatory element justifies the expectation of these effects and which conditions of success must be considered?	What relations between aspects of the learning content must be considered? In which refined learning trajectory (or learning landscape) can the relevant aspects of the learning content be structured in order to increase access for all participants?

Table 2: Typical theory elements in didactical design research and their different functions (Prediger, 2019a, p. 14)

For each of these components in this if-then structure, constructs are required, and the arguments usually use descriptive, explanatory, or other prescriptive theory elements. Due to the complexity in which design principles relate different theory elements, McKenney and Reeves (2012) emphasized: "'design principles' is probably the most prevalent term used to characterize the kind of prescriptive theoretical understanding developed through educational design research … [as they] integrate descriptive, explanatory and prescriptive understanding to guide the design of interventions" (p. 35). This is why the focus on argumentative grammars should be widened. Even if these "how" questions might be researched in a learning environment with a particular topic, this topic is not necessarily in the center of the theorizing.

In contrast, *content-related didactical design research* can be characterized as that branch of design research in which the theorizing on "how" questions is complemented, substantiated, and refined by theorizing on "what" questions (see Gravemeijer & Cobb, 2006; Prediger, 2019a, 2019b). Conjectures about empirically grounded hypothetical *learning trajectories* (Bakker, 2018) can now be conceived as a complex prescriptive theory element composed of various other ones: The research often starts with a vague idea of the learning content, which becomes increasingly concise while analyzing participants' learning pathways (van den Heuvel-Panhuizen, 2005). In these cases, constructs, descriptive, and explanatory theory elements are generated by systematically contrasting the intended perspectives on subject-matter aspects to the participants' individual perspectives, and the content is restructured, with other priorities, starting points, and connections. Formulating the intended learning trajectory (prescriptive theory element) and capturing participants' individual learning pathways (descriptive and explanatory theory elements) requires detailed constructs in order to articulate both on a micro level. In the beginning, the constructs are often humble, then successively refined in the iterative cycles.

Table 2 shows typical "how" and "what" questions and the different functions of theory elements (in an order as often applied in design research projects). While research processes are never linear with respect to the working areas addressed, the direction of theorizing often starts from propositional theory elements including (perhaps still humble or implicit versions of) normative theory elements and humble prescriptive heuristics and then elaborates them by iteratively refining and connecting categorial, descriptive, and explanatory elements. The *refined prescriptive elements* are considered to be the major outcome, as they condense the other elements.

3.3 Enriching the argumentative grammars by making explicit other theorizing processes

The list of unfolded logical structures of involved theory elements in Table 2 can help to unpack the conjectures discussed in Section 2: Confrey (2006) implicitly referred to nearly every kind of theory element for "what" questions, and Sandoval (2014) to nearly every kind of theory element for "how" questions. Cobb et al. (2017) combined the "what" and the "how" questions, with a focus on explanatory and refined prescriptive theory elements drawing upon constructs. Whereas the discussed argumentative grammars mainly focus on empirical warrants and backings for the connection of data and claims for the prescriptive theory elements, they do not account for the theorizing processes needed to gain suitable constructs, meaning the normative elements and explanatory elements backing the prescriptive conjectures. These theorizing processes involve, for example (Prediger, 2019a):

- *Identifying* an interesting phenomenon and developing constructs for describing and explaining it
- *Refining* constructs in order to increase their explanatory power

- *Connecting* two descriptive elements to explanatory elements
- *Transforming* an explanatory theory element into a conjecture for a prescriptive theory element
- *Refining* a prescriptive theory element by adding conditions
- *Testing conjectures:* In the first cycles, the testing and iterative refinement of conjectures is done by qualitative research; only when the conjectures are sufficiently refined, they can also be strengthened by standardized instruments and larger samples in quantitative research.

Sandoval (2014) described the process of conjecture mapping as one to *make implicit ideas of the design researchers more explicit.* He emphasized the relevance of identifying the theoretically salient aspects (*constructs and explanative theory elements*) and the refinements needed in the next iteration of the conjecture map (*in constructs and explanative theory elements*, in the *normative elements* of mediating processes, and in refined *prescriptive* conjectures). His illustrative examples show refinements consisting of adding design characteristics and relevant mediating processes. Cobb et al. (2017) put emphasis on the theorizing process of *normatively* specifying learning goals that can rarely simply refer to existing curriculum documents, but involve a reconceptualization of learning goals within the given theoretical framework. This can involve concisely articulating content-specific constructs for learning goals (*constructs* and *explanative* elements about their connections).

In total, the distinction of theory elements with different functions and structures helps to articulate more concisely the complex processes involved in theorizing in design research with its iterative interplay of theorizing, design, and empirical work.

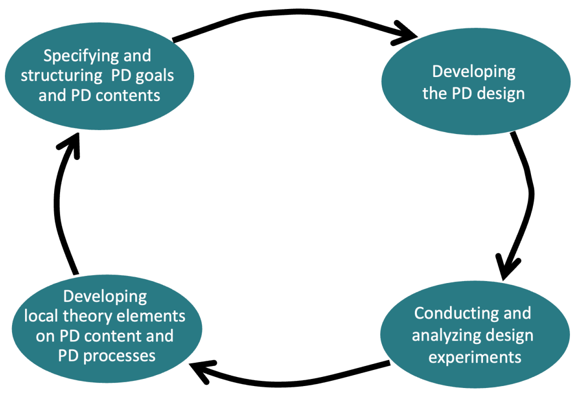
4.0 Theorizing as refining categorial and explanatory theory elements before conjecturing: An illustrative case from a PD design research project

In this section, an illustrative case of a PD design research study is presented in order to show how far theorizing processes of refinement of constructs and of explanatory elements are required to go before more complex design conjectures can be composed.

4.1 Lifting methodological and meta-theoretical considerations from classroom design research to PD design research

Although most design research studies continue to be conducted at the classroom level, design research approaches have been increasingly lifted to pre-service teacher education (see scoping review by Lasthein Lehrmann et al., 2022) and in-service professional development (Cobb et al., 2017; Prediger et al., 2016).

Figure 3: Working Areas in content-related PD Design Research (Prediger, 2019)



In these cases, the learners are not students but teachers. The PD learning content might comprise not only subject-matter content knowledge but also pedagogical content knowledge and often target instructional practices in which teachers are invited to engage. Like classroom designs, PD designs can be materialized in PD resources, such as worksheets for PD activities, videos to be discussed, and slides or videos for input, and can come with various participant structures and discourse practices (Sandoval, 2014). We therefore also decided to lift our content-related design research approach to the PD level, with its four working areas, which are not necessarily sequenced from top left to button left (see Figure 3).

Content-related design research approaches for professional development are particularly promising for filling two important research gaps (Prediger et al., 2016): In their scoping review, Goldsmith et al. (2014) called for a stronger *process focus* of PD research, noting that "existing research tends to focus on program effectiveness rather than on teachers' learning" (p. 21). Garet et al. (2016) called for a stronger *content focus* by emphasizing a "need to improve our understanding of the aspects of knowledge and practice that effective teachers should master" (pp. 9-10). The example in this section can exemplify how content-related PD design research contributes to a deeper focus on both process and content.

With regard to *theorizing* in content-related design research, Cobb et al. (2017) suggested shifting the theorizing processes of conjecturing to the PD level by describing typical theoretical outcomes and argumentative grammars for PD design research as involving "demonstrating that the participants would not have developed particular forms of practice but for their participation in the design study" (Cobb et al., 2017, p. 222), the "conjecturing about envisioned teacher learning trajectories" (p. 221), and "identifying the specific aspects of the PD learning environment that were necessary … [for] the emergence of these successive forms of practice" (p. 222). Note that, like on the classroom level, generating conjectures and identifying specific support here means it is conducted within qualitative, hypothesis-generating modes, not by validating hypothesis quantitatively, such as, for example, in randomized controlled trials.

We will show an example in which the articulation of intended forms of practices and the underlying refined learning outcomes is key, so theorizing also involves elaborating categorial, descriptive, and explanatory theory elements.

4.2 Background of the project Mastering Math with its PD design research studies

The illustrative case stems from the long-term project Mastering Math, which started in 2007. The overall goal of the Mastering Math project has been to empower mathematics teachers for their work with Grade 3-7 students who are at risk of being underserved and have not yet developed understanding of basic concepts in arithmetic (in brief, *at-risk students*). Material support is provided for teachers through teacher manuals and Mastering Math curriculum materials, in 45 modules for formatively assessing and enhancing students' understanding for basic concepts such as place values and meaning of multiplication and division (presented in Prediger et al., 2019). The curriculum materials serve as the base of a task-based PD program (Swan, 2007) accompanying teachers' pathways of action and reflection while experimenting with the materials (Clarke & Hollingsworth, 2002).

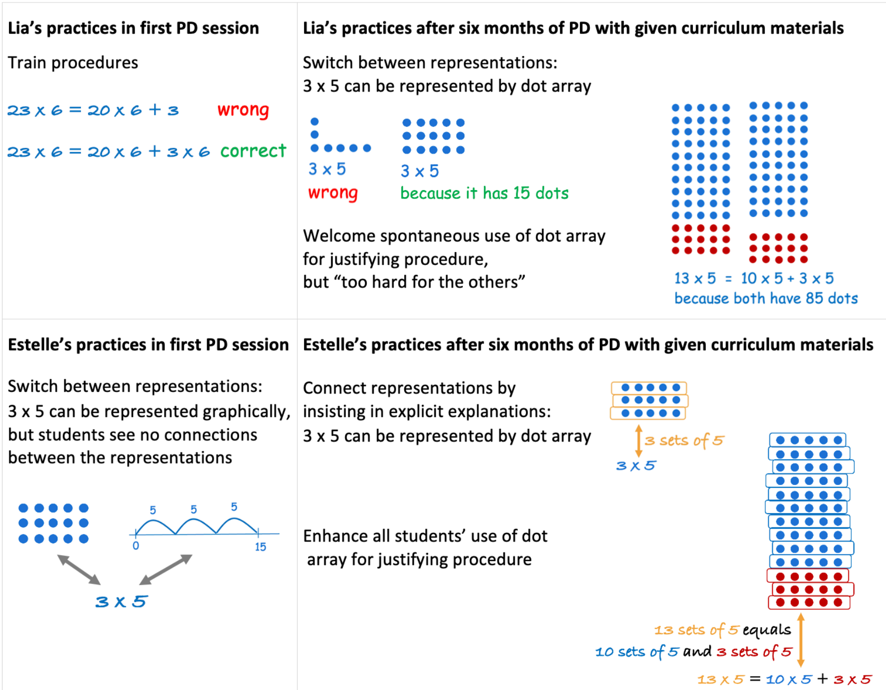
The PD design was iteratively developed in several cycles in which we empirically investigated what kind of learning opportunities teachers needed, first in informal action research modes and later in more rigorous modes of PD design research with methodologically more controlled methods of data gathering and data analysis. During the project, theoretical accounts of teachers' professional growth in the PD content were successively developed in various areas (Prediger, 2024) and informed the research-based redesign of the PD modules. Meanwhile, quantitative empirical evidence was provided for the effectiveness for student learning (Prediger et al., 2019) and teacher learning (Prediger et al., 2023), the latter of which not having been possible in earlier cycles because the learning goals to be measured first had to be specified in depth (Sandoval, 2014).

4.3 An undertheorized vignette from a PD design research project

In this section, an undertheorized vignette from the third design experiment cycle is presented to serve as an illustrative example of the needs and procedures of theorizing to make sense of the processes of professional growth initiated in a PD program and to develop a language for making targeted design decisions. In the beginning, the phenomena are often not well articulated, like in this vignette.

Lia and Estelle were mathematics teachers and both highly dedicated to supporting their at-risk students, so they participated in the Mastering Math PD program. Estelle entered the program because she was unsatisfied with her teaching for at-risk students, saying in the first meeting: "I do not succeed in making these kids really understand. For every representation, we start anew, students see no connections" (translated by SP).

Figure 4: Lia's and Estelle's changes in mathematical emphasis after three PD sessions and 6 months of experimenting with the Mastering Math curriculum materials



Lia entered the program with higher satisfaction about her teaching, but discontent with students' abilities to remember what they had learned: "We invest a lot in training multiplications such as 23 x 6. But after three weeks, my low achievers make again 20 x 6 + 3. They simply forget too much and too quickly."

After a year in the Mastering Math PD program and experimenting with the Mastering Math curriculum materials, multiplication was one of the last areas of content we worked on. Lia reported proudly: "Only through these diagnostic tasks I have realized that many of the kids don't know the meaning of multiplication. I invested a lot in making all of them draw full dot arrays, not only 3 dots in vertical and 5 dots in an L-form, because we must see all 15 dots. With the dot array, one student surprised me by arguing *why* 13 x 5 must be 10 x 5 + 3 x 5. Of course, justification was too hard for the others." Figure 4 depicts the representations she referred to. Estelle reported on the same curriculum material: "Yes, indeed, I have seen the same problems! When we worked on it, I helped students to explain how the dot array corresponds to 3 x 5, with the "three sets of fives"; they could also find the same structure in the number line. And with my constant insisting in these explanations, they succeeded also in justifying why 13 x 5 can be calculated by 10 x 5 + 3 x 5." Both teachers substantially developed their teaching practices, yet Estelle had started at a different point and came further. In the beginning, however, we were not able to describe and explain the phenomenon more explicitly.

This vignette (and many similarly vague experiences in the first PD design experiment cycles) exemplifies the need to clarify many questions, some of them for researchers in general and some in particular for the design researchers engaged in the PD design of the Mastering

Math PD program. Some examples of questions that were raised are listed below:

Descriptive questions

- What characterizes the teachers' instructional practices and their expertise at the beginning and at the end of the PD program, and how typical are these for many teachers?
- What elements of the offered PD content and support materials do they pick up (and what is filtered out)?

Explanative questions

- What underlies the teachers' instructional practices, and what influences their decision making?
- Why do teachers pick up certain PD content elements and filter out others?
- What are the underlying mechanisms of professional growth that explain the changes in teachers' practices?

Normative questions

- What exactly are appropriate and realistic PD content goals, and how can they be justified?

Prescriptive consequences

- How exactly should the PD design be refined to provide more focused learning opportunities for the specified unpacked PD content goals?
- What PD design characteristic can lead to which mediating process and to approaching a certain PD content element?

The questions indicate that theorizing for PD design research is even more complex than on the classroom level, as the learning content is more complex and subtle. So we first had to develop a framework in which to articulate the teachers' practices and professional growth.

4.4 Theoretical framework for describing and explaining the teachers' practices and professional growth

To treat these questions in systematic ways, it is essential to draw upon existing theoretical frameworks and models that provide a language for describing phenomena and explaining their connections in two major areas: the expertise of teachers with their instructional practices and typical mechanisms of professional growth. In this section, two models for both areas are introduced and integrated into our framework.

4.4.1 Model of content-related teacher expertise or specifying relevant PD content elements underlying teachers' practices

The ultimate goal of many PD programs has been to develop teachers' instructional practices, along with the underlying knowledge categories and orientations. This articulation of the overall PD goal combines two perspectives that were formerly considered complimentary yet often unconnected: *a situated perspective* on teachers' classroom practices with *a cognitive perspective* on underlying knowledge categories and orientations (Depaepe et al., 2013). The model we chose for our theoretical background theory should provide constructs to capture practices and the underlying knowledge categories and orientations and to provide explanatory power on how they are connected.

We found an appropriate general framework in the work of Bromme (1992), who characterized teachers' instructional practices as ways to cope with typical situational demands in classroom situations and analyzed the underlying categories of perceiving and thinking. Similarly, Schoenfeld (2010) offered a framework in which teachers' decision making about instructional practices is explained by teachers' orientations. Both authors also mention the pedagogical tools. Synthesizing the constructs of both authors, Prediger (2019b) defined several *constructs* of the *Model of Content-Related Teacher Expertise* as follows:

- *Jobs*: Typical and often complex situational demands that teachers have to master in classrooms (in each PD project, restricted to the jobs of relevance for the PD content in view).
- *Practices*: Recurrent patterns of teachers' utterances and actions for managing the jobs. Teachers' practices can be characterized by the underlying categories, pedagogical tools, and orientations upon which teachers implicitly or explicitly draw:
    - *Pedagogical tools*: Tangible or visible tools applied to manage the jobs (e.g., enacted facilitation moves, assessment tasks, manipulatives, or other didactical artifacts).
    - *Categories*: Categorial knowledge elements that filter and focus teachers' perceiving and thinking. Although generic pedagogical knowledge provides relevant categories too, we focus here on the pedagogical content knowledge (PCK) categories that teachers explicitly or implicitly chose as their filters for perceiving and thinking.
    - *Orientations*: Generic or content-related beliefs and pedagogical attitudes about mathematics and its teaching and learning that implicitly or explicitly guide the teacher's perception and prioritization of jobs (see Schoenfeld, 2010, p. 29).

These constructs are highly functional to *describe* teachers' decision-making (similar to Schoenfeld, 2010), and the theoretical framework provided by Bromme (1992) allows *explaining* the interplay between practices and the underlying knowledge categories, orientations, and used pedagogical tools.

While Bromme (1992) phrased his framework of teacher expertise in generic terms, Prediger (2019b) further developed it into a conceptual model that allows design researchers to specify and describe teacher expertise *of a certain area of PD content*. This search was inspired by the research synthesis of Goldsmith et al. (2014), with their call for more content-specific PD research unpacking various areas of PD content.

The *Model of Content-Related Teacher Expertise* can be used to determine which categories and orientations are needed or already used by teachers. In a *descriptive mode*, the practices of teachers are observed and analyzed with respect to the pedagogical tools, categories, and orientations they use for managing certain jobs. Complementarily, in a *prescriptive mode*, PD design researchers prescriptively determine the pedagogical tools, categories, and orientations expert teachers *should* use, for instance, by analyzing expert practices (as Bass & Ball, 2004, suggested in their job analysis). The pathway from the current to the intended expertise can be unpacked by specifying the necessary orientations for teachers' practices (both current and intended) to cope with the jobs and by identifying the underlying categories that should or do guide their practices.

4.4.2 Slightly adapted Model of Professional Growth

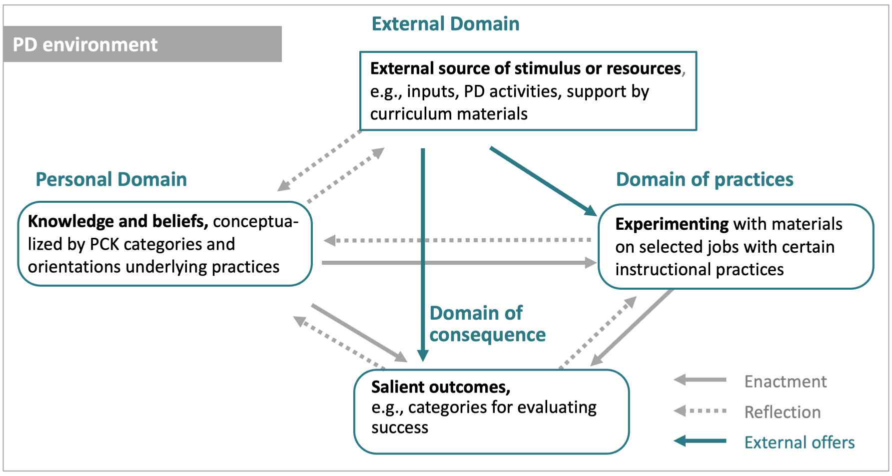
For explaining and promoting teachers' pathways in professional development (PD), the theorizing in the Mastering Math project draws upon the well-established *Interconnected Model of Professional Growth* by Clarke and Hollingsworth (2002), which has been widely used not only for describing and explaining, but also for designing PD that promotes professional growth in modes of action and reflection. This subsection follows Prediger (2024) in presenting and adapting the model.

The model includes four analytic domains: the *external domain* (with external sources of information, stimulus, or support materials for teachers), the *personal domain* (teachers' knowledge or attitudes), the *domain of practice* (in which experimentation with practice can take place), and the *domain of consequence* (with salient outcomes such as students' learning gains). The model identifies different mechanisms by which change in one domain can be associated with change in another. Rather than claiming simple mechanisms of transmission from the *external domain* via changing the *personal domain* to the *domain of practice* and then to the *domain of consequence*, they emphasize an "interconnected, non-linear structure" between these domains and identify different "particular 'change sequences' and 'growth networks,' giving recognition to the idiosyncratic and individual nature of teacher professional growth" (Clarke & Hollingsworth, 2002, p. 947).

An example of such an individual change sequence might be teachers experimenting in the domain of practice, monitoring students' thinking in the domain of outcomes, and thereby expanding their knowledge about student thinking in the individual domain, which is then the result of the change sequence rather than its start.

In Figure 5, a slightly adapted model is depicted (adaptions outlined in Prediger, 2024) in which the constructs of the Model of Content-Related Teacher Expertise have been integrated systematically in the respective domains, which is possible as the underlying frameworks are compatible in the combination of situated and cognitive perspectives (Bromme, 1992; Depaepe et al., 2013). The personal domain can also be a collective domain in case of PD taking place in collaborative groups (Prediger, 2024).

Figure 5: Interconnected Model of Professional Growth (adapted from Clarke & Hollingsworth, 2002)



4.5 Substantiating the model for the PD content fostering at-risk students' understanding of basic concepts in the Mastering Math PD design research project

Bromme's (1992) general framework and the suggested integrated model in Figure 5 gain their explanative power for content-related purposes when filled in content-specific ways. This is the key idea of the Model of Content-Related Teacher Expertise (Prediger, 2019b): To explain teachers' practices and professional growth for a particular PD content, we identify the typical jobs related to an area of PD content (in our case fostering at-risk students' conceptual understanding of basic concepts) and the content-related pedagogical tools, orientations, and filtering categories that underlie the practices of teachers in the particular area of mathematics education that is the content focus of the PD (Prediger, 2019b).*Theorizing that informed the PD design*

When starting with the Mastering Math project, we were already aware that monitoring students' learning progress and enhancing students' understanding are crucial jobs for mathematics teachers working with at-risk students (Slavin & Madden, 1989). We developed the curriculum materials with formative assessment tasks and enhancement tasks for learning trajectories in order to support teachers' practices with pedagogical tools that we can provide as external sources (Prediger et al., 2019). This decision was fueled by a

- *normative theory element:* Mathematics teachers in the Mastering Math project should learn to monitor and enhance students' conceptual understanding. This overall PD learning goal is justified by empirical research on the classroom level showing that understanding the basic concepts is crucial for at-risk students' further mathematical learning progression (Slavin & Madden, 1989; Gersten et al., 2009).

From earlier empirical research on teachers' practices (Zohar et al., 2001), we drew the relevance of a conceptual rather than procedural orientation, so this construct helped to articulate an

- *explanative theory element:* Many teachers do not enact conceptual enhancement practices because their conceptual orientation is not very strong.

From there, we derived a

- *prescriptive design conjecture:* If we want to achieve teachers' adoption of conceptual enhancement practices, we need to enable them to also have good experiences in enhancing conceptual understanding for at-risk students (Zohar et al., 2001).

Later on, we realized that this included another task of teaching – specifying learning goals (Morris et al., 2009) – and that teachers can be supported in setting conceptual learning goals by conceptual formative assessment tasks (Swan, 2007). This led to the second

- *prescriptive design conjecture:* If we want to achieve teachers' adoption of conceptual enhancement practices, we need to make sure that they set conceptual learning goals. This can be supported by conceptual formative assessment tasks, because in the Interconnected Model of Professional Growth, assessment tasks help to treat conceptual understanding as a salient outcome.

The categories that teachers explicitly or implicitly use and should use for thinking and perceiving must be further specified (in a prescriptive mode informing the PD content goals). From the beginning, we were aware of teachers' PCK categories for articulating the overall learning goals in understanding basic concepts, such as understanding place values and meaning of the multiplication (Gersten et al., 2009). In later PD design experiment cycles, we have learned that a second sub-job was critical for specifying the learning content, namely, unpacking these basic concepts into their constituent parts, here called content elements (see Morris et al., 2009).

Theorizing for the PD content: Teachers' practices and underlying categories

With the introduced theoretical constructs, we have now the categorial means to analyze the undertheorized introductory vignette from Section 4.3 to show how the constructs are instrumental to *describe* and *explain* Lia's and Estelle's growth.

The analysis is summarized in Table 3 and reveals which orientations and categories were initially missing but were needed to explain Lia's growth. Hence, it helped to specify the PD content in more detail (*refining the normative theory elements based on descriptions connected to explanative theory elements*).

In the first PD session, Lia reported on training procedures and students forgetting them (see Figure 4). From her report about monitoring procedures without errors, we infer that she set procedural learning goals.

	Lia's practices in the first PD session	Lia's practices after 6 months in the PD program
Practices	Set procedural learning goals Monitor correctness of procedures Foster simply by training	Set procedural and conceptual learning goals Monitor task completion Foster by supporting task completion
Pedagogical tools	Procedural training tasks	Assessment tasks and enhancement tasks with multiple representations
Underlying categories of thinking and perceiving ex-/implicitly used	Categories for learning goals: • Procedure without errors Success category: • Task completion • Forgetting	Categories for learning goals (not yet further unpacked) • Dot arrays as the relevant representation • Procedure without errors and justified Success category: • Task completion
Underlying orientations	Procedural orientation Short-term orientation	Procedural and conceptual orientation Short-term orientation
	Estelle's practices in first PD session	**Estelle's practices after 6 months in the PD program**
Practices	Set conceptual learning goals Monitor overall learning progress Foster by enhancement of understanding	Set conceptual learning goals Unpack tgoals into content elements Monitor learning of unpacked content elements Foster by focused enhancement towards unpacked content elements
Pedagogical tools	Enhancement tasks with multiple representations	Assessment tasks and enhancement tasks with multiple representations, moves for supporting explanation
Underlying categories of thinking and perceiving ex-/implicitly used	Categories for overall learning goals: • Understanding meaning of multiplication • Dot array and number line as relevant representations Success category: Learning progress	Categories for unpacked goals into content elements: • Dot array and number line as relevant representations • Multiplication as counting in units • Explanation of units • Procedure justified Success category: Learning progress in targeted elements
Underlying orientations	Conceptual orientation Long-term orientation	Conceptual orientation (also understand procedures) Long-term orientation

Table 3: Describing and explaining Lia's and Estelle's growth in self-reported practices with their underlying categories and orientations

Her main pedagogical tools were procedural training tasks, so her fostering practices were restricted to training procedures without further learning opportunities. In total, her practices seemed to be consistently shaped by a strong procedural orientation.

After three PD sessions and six months of experimenting with the given curriculum material, Lia reported widening her practices, setting also conceptual learning goals and monitored whether students could complete the new conceptual tasks. In her fostering practices, she supported students to correct the solution to represent 3 x 5 by an L-form (see Figure 4), but only by referring to the total of 15 dots. This support allowed a shallow switch between representations (i.e., the completion of the task) without enhancing students' deeper understanding of multiplicative structures. She widened her purely procedural orientation using aspects of a conceptual orientation, but her surprise that a student justified the procedure with the dot array and her decision not to include the justification as a learning goal for all students indicated that she still fluctuated between procedural and conceptual orientations.Estelle, in contrast, already held a strong conceptual orientation prior to the PD, as she set conceptual learning goals, monitored students' overall conceptual learning progress, and enhanced students' understanding of the meaning of multiplication by using tasks for switching between symbolic and graphical representations (dot array and number line), while neglecting procedures. With these more ambitious learning goals in mind, she evaluated her teaching more critically than Lia. These different practices can be *explained* by Estelle's different monitoring category: She monitored students' learning progress, not only task completion.

Theorizing the processes of professional growth

In the PD sessions preparing the teaching experiment for enhancing students' understanding of the meaning of multiplication, Estelle was attentive to the unpacking of this overall learning goal into its constituent content elements: the counting in units and the language for explaining how the dot array is connected to the multiplication by "three sets of fives" (Götze & Baiker, 2021). When reporting from her classroom trials later on, Estelle reported that she could also enhance students' learning progress in these detailed content elements, so that learning progress in these unpacked content elements served her as fine-grained success categories. Her conceptual orientation had been widened to the unpacked learning goal of procedures to be justified by unit structures in dot arrays, and she reached this new learning goal for all (not only some selected) students by pressing for explanations for how the representations are connected.

This analysis of Lia's and Estelle's growth patterns illustrates how the introduced *constructs* of the Model of Content-Related Expertise can be used to *describe* changes in practices and *explain* them by the underlying changes of orientations and categories of perceiving and thinking. The analysis also draws upon the *explanatory* theory elements inherent in the substantiated Model of Professional Growth (see Figure 5) to account for the *interplay* of offered support in the external domain, changes in practices in the domain of practice, and changes in the underlying orientations and PCK categories for articulating learning goals in the personal domain.

The analysis was further deepened to explain also why Lia (unlike Estelle) was not ready to pick up the professional learning opportunities for unpacking the conceptual learning goals into relevant content elements (the researcher's theorizing process is documented for another case study in Prediger, 2024):

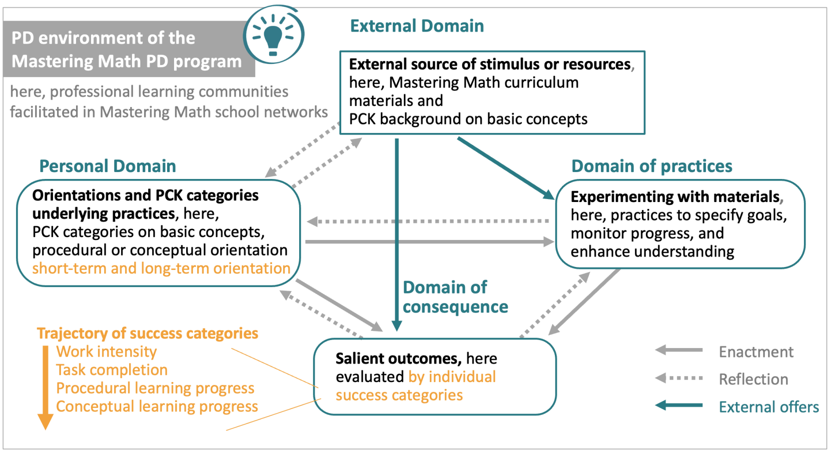
A key construct in the Interconnected Model of Professional Growth (Clarke & Hollingsworth, 2002) that helps to explain Lia's decisions is the *salient outcome* in the domain of consequences: When evaluating the success of certain instructional practices, teachers consider salient outcomes in students' behavior. The contrast between the two teachers' growth pathways indicates how this evaluation is shaped by personal success categories that determine what is a success in the individual teachers' perspectives. In line with the PD program agenda, Estelle evaluated her teaching outcomes by considering students' individual learning progress. Her decisions about how to foster students' understanding were informed by this feedback loop and helped her to develop her enhancement practices. In contrast, Lia evaluated her instructional practices by considering whether her students could complete the task without errors, but not by considering their learning progress. For example, she supported students to switch between 3 x 5 and the dot array by simply referring to the needed 15 dots. This support can bring all students to the completing dot array in the workbook, but not necessarily to constructing meanings of multiplication as counting in units.

The *success category of task completion* instead of learning progress has been empirically identified in various case studies within the Mastering Math project and kindred projects (Prediger, 2023, for an overview) and also by other researchers (Watson & de Geest, 2005; Herbst, 2003). It comes with a second relevant orientation that we termed short-term rather than long-term orientation (Prediger et al., 2016). In various contexts and for different jobs, we could show that in short-term orientations, other practices turn out to be more rational than in long-term orientations (Watson & de Geest, 2005; Prediger et al., 2016). For example, many fostering practices reveal immediate support optimized for making students complete the task (we call them compensation practices, Prediger et al., 2023) that substitute shallow support, which is not adaptive to the real learning needs, for the deep learning opportunities necessary in real enhancement practices.

Abstracting from cases to general pattern of PD growth progression

With multiple case study analyses from different cycles similar to the one presented here, we identified a PD growth progression on success categories that is depicted in the *Mastering Math Model of Professional Growth* in Figure 6, for which the orange parts were only added after the iterative refinement (Prediger, 2023, 2024).

Figure 6: Mastering Math Model of Professional Growth (orange parts after iterative refinement)



4.6 Validating conjectures by quantitative research

Only after successively developing, refining, and integrating the theoretical constructs and descriptive and explanatory theory elements in a long process of iterative refinements were we able to formulate well-articulated hypotheses that could also be operationalized with standardized measures and tested in quantitative evaluation studies:

Overall Design Conjecture: With a PD design according to the conjecture map in Figure 6, teachers' practices and orientations can be developed towards the targeted practices and underlying categories.

This rather holistic conjecture (corresponding to those in Figure 1) was tested by an evaluation study in a pre-post design. Standardized (partially vignette-based) instruments for capturing teachers' self-reported and simulated practices were administered in the first and sixth PD session of a one-year PD program. The evaluation revealed that indeed, the participating teachers (n = 95) significantly developed their practices and underlying orientations (study reported in more detail in Prediger et al., 2023), but with one exception: While the short-term orientation became less relevant for goal-setting and monitoring practices (participants reported going back more often to basic learning content before treating the official actual content), the self-reported fostering practices were shaped more by short-term compensation than by long-term orientated enhancement practices for deep learning. This challenge must be treated in the next PD cycle.

Conjecture on particular PCK categories: When the PD design puts particular emphasis on PCK categories for unpacking the learning content, teachers can better develop focused enhancement practices.

This much more focused design conjecture (corresponding more to the argumentative grammar in Figure 2) was tested in a randomized controlled trial in pre- and post-test control group design, focusing only on a particular design element examining the ways in which we put emphasis on particular PCK categories for the meaning of multiplication (Wischgoll & Prediger, submitted). Mathematics teachers (n = 94) participated in a 2-hour PD session on unpacking and enhancing students' understanding of multiplication (36% of whom used only practices aiming at task completion, like Lia). The treatment group received a much more explicit systematization focusing the multiplication as counting in units and the language of units for explaining the connection between representations. Teachers in this treatment group conducted significantly more targeted enhancement practices in the vignette-based test than the control group without explicit systematization.

5.0 Summary on processes of theorizing in PD design research

Section 4 presents illustrative insight into the complexities of theorizing on the PD level for which the learning content is more complex to capture than on the classroom level. In Table 4, some of the most important theory elements generated and substantiated in the Mastering Math PD design research project are summarized.

	Examples from general theoretical models	Content-specific substantiation for the PD content in the Mastering Math project
Descriptive theory elements with relevant constructs	Model of Content-Related Teacher Expertise: Teachers' practices for certain jobs can be characterized by the pedagogical tools and underlying orientations and categories of perceiving and thinking.	Expertise for the PD content fostering students' understanding is expressed in productive practices for specifying, monitoring, and enhancing students' understanding of basic concepts.
Normative theory elements	PD program aims at teachers' growth in productive practices for all relevant jobs. Empirical evidence for the productivity of certain practices is to be given from classroom research.	Mastering Math aims at teachers' growth in productive practices for *specifying, monitoring, and enhancing students' understanding of basic concepts.* Empirical evidence for the productivity was provided in classroom field trials.
Explanatory theory elements on expertise	Unproductive orientations can hinder productive practices. Limited categories can hinder productive practices.	A *procedural rather than conceptual orientation* can hinder productive practices for specifying, monitoring, and enhancing students' understanding of basic concepts. Refined *PCK categories* are needed to unpack the mathematical learning goals.
Refined normative theory elements	To reach the PD program goals of specified productive practices, the PD learning opportunities should aim at promoting productive orientations and extended, fruitful categories.	To enable teachers to conduct productive practices for *specifying, monitoring, and enhancing students' understanding of basic concepts,* a conceptual orientation should be developed and *PCK categories* (involving all concept elements of, e.g., multiplication) should be refined to unpack the mathematical learning goals.
Humble prescriptive heuristics	When certain PCK categories are extended and curriculum materials are provided that allow successful inquiries of new practices with these PCK categories, teachers can develop their practices. The reflection of the experiences is key for developing orientations.	When curriculum material is provided for understanding basic concepts and the relevant content goals in assessment and enhancement are unpacked, teachers can appropriate these unpacked content goals as PCK categories and develop their monitoring and enhancement practices. The reflection of students' conceptual learning can lead to developing conceptual orientations and consolidating the PCK categories being activated.
Explanatory elements on professional growth	The teachers' success categories in the domain of consequences influences strongly the individually felt need to change practices (or not).	A *short-term rather than long-term orientation* can hinder productive practices as it prioritizes the *success category of task completion* before success categories on students' learning progress.
Refined prescriptive theory elements	In order to change certain orientations, salient classroom experiences need to be made accessible for teachers on the relevance of the orientation.	Still in need to be substantiated in further cycles.

Table 4: Summary of theory elements generated and substantiated in the Mastering Math PD project

While the table might suggest that the general theoretical models were already available and only needed to be substantiated for the particular PD content in view, it must be emphasized that the Model of Content-Related Teacher Expertise and its integration into the Model of Profes-sional Growth were also developed during this 10-year design research process. For the next project, theorizing became easier as it could concentrate on content-specific substantiations, which can also be hard work, as Garet et al. (2016) articulated.

Overall, the author hopes to have provided a helpful illustration of the complex interplays be-tween different theory elements and the different processes of theorizing that all need to take place while articulating and before testing conjectures.

Section 4 also illustrates how the Mastering Math PD project extended typical limitations of design research by combining design research with subsequent quantitative studies (Prediger et al., 2019; Prediger et al., 2023, Wischgoll & Prediger, submitted).

Summing up, design research results in more than local problem solutions when it contributes to empirically grounded theorizing. Although key theoretical contributions can be prescriptive conjectures (Sandoval, 2014; Bakker, 2018; Cobb et al., 2017), these conjectures rest upon categories and descriptive, explanative, and normative theory elements, and the more intense the components are connected, the deeper the theorizing can be. Whereas all design researchers are challenged by the complex interplays of all these elements, PD design research is particularly challenging as the areas of PD content and the growth processes are highly complex and require thorough unpacking. We hope to continue the methodological discourse in the future to provide a successively elaborate metatheoretical language to account for all (often implicit) theorizing processes.

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